II. **Solution by J. A. CAPARO, C. E., Notre Dame University, Notre Dame, Ind.**

Let α=one-half the vertical angle of cone, a, and an radii of spheres. With the vertex of the cone as origin, and the axis of y as the axis of the cone, the mass of the frustum contained between the spheres is

$$M=k\pi\tan^2\alpha\int_{a\cos\alpha}^{na\cos\alpha} y^3dy+k\pi\int_{an\cos\alpha}^{an}(a^2-y^2)ydy-k\pi\int_{a\cos\alpha}^{a}(a^2-y^2)ydy$$

$$=\frac{\pi ka^4(n^4-1)\sin^2\alpha}{4}.$$

These integrals represent the mass of a frustum whose bases are planes passing through the line of intersection of the surfaces, and the masses of the spherical segments added and subtracted from the above mass. Call these masses M_c, M_a, M_n, and $\bar{y}_c$, $\bar{y}_a$, $\bar{y}_n$, distances of their centroids from origin.

Then, $M_c\bar{y}_c=\pi k\tan^2\alpha\int_{a\cos\alpha}^{na\cos\alpha} y^4dy=\frac{1}{5}\pi ka^5\cos^5\alpha\tan^2\alpha\,(n^5-1)$.

$$M_a\bar{y}_a=\pi k\int_{a\cos\alpha}^{a}(a^2-y^2)y^2dy=\frac{\pi ka^5}{15}(2-5\cos^3\alpha+3\cos^5\alpha).$$

$$M_n\bar{y}_n=\pi k\int_{an\cos\alpha}^{an}(a^2-y^2)y^2dy=\frac{\pi ka^5n^3}{15}[5(1-\cos^3\alpha)-3n^2(1-\cos^5\alpha)].$$

But $M\bar{y}=M_c\bar{y}_c+M_n\bar{y}_n-M_a\bar{y}_a$.

$$\therefore \bar{y}=\frac{4a[2+5n^3-3n^5+n^3\cos^3\alpha(3n^2+5)+2\cos^3(4+3\cos^2\alpha)]}{15(n^4-1)\sin^2\alpha}.$$

227. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

Solution by the PROPOSER.

Let $AB=2a$ be the tunnel; $OA=OB=R$; C the mid-point of AB; P and R, points between A and C; $OP=y$, $PC=x$, $CR=d$, $\angle POC=\theta$; f the acceleration at P; f' the acceleration along PC at P, and R the starting point of the sphere. Beneath the earth's surface, the acceleration varies directly as the distance from the center.

$\therefore f:g=y:R$ or $f=gy/R$. $f':f=\sin\theta:1$ or $f'=f\sin\theta=gy\sin\theta/R$.

But $y=x/\sin\theta$. $\therefore f'=gx/R$.

I. If the tunnel is perfectly smooth, the equation of motion is $d^2/dt^2 + gx/R = 0$. $\therefore (dx/dt)^2 + gx^2/R = C$. Since $t=0$ when $x=d$, $C = gd^2/R$.

$\therefore (dx/dt)^2 = (g/R)(d^2 - x^2) = (\text{velocity})^2 = v^2$. When the sphere arrives at C, $x=0$. $\therefore v = d\sqrt{(g/R)} = a\sqrt{(g/R)}$. When the sphere starts at A,

$$t = \sqrt{\frac{R}{g}} \int_0^m \frac{dx}{\sqrt{(d^2 - x^2)}} = \sqrt{\frac{R}{g}} \sin^{-1}\frac{m}{d}. \quad \text{If } m=d,\ t = \frac{\pi}{2}\sqrt{\frac{R}{g}}.$$

As this expression is independent of the distance from C, the time for the sphere to move from A to C or from any point between A and C to C is the same. The time is the same for any tunnel, and is the same as the time required for the sphere to fall from the surface to the center, or from any point beneath the surface to the center of the earth.

Let $R = 20902410$ feet, $g = 32.10614$ feet.

$\therefore \sqrt{(R/g)} = 806.871$, and therefore $t = (\frac{1}{2}\pi)\sqrt{(R/g)} = 1267.433$ seconds $= 21$ minutes, 7.433 seconds.

II. When the tunnel is perfectly rough. Let F be the friction. Then the equations of motion are $d^2x/dt^2 + gx/R + F = 0$, $k^2 d^2\phi/dt^2 = bF$, $b\phi = x$ or $bd\,\phi = dx$, $bd^2\phi = d^2x$.

$$\therefore \left(\frac{k^2}{b^2}\right)\frac{d^2x}{dt^2} = F;\ \therefore \left(\frac{b^2+k^2}{b^2}\right)\frac{d^2x}{dt^2} + \frac{gx}{R} = 0;\ \frac{d^2x}{dt^2} + \frac{5gx}{7R} = 0;\ \left(\frac{dx}{dt}\right)^2 = v^2 = \frac{5g}{7R}(d^2 - x^2).$$

$\therefore v = d\sqrt{(5g/7R)}$, at the point C; $v = a\sqrt{(5g/7R)}$, when the sphere starts at A.

$$t = \sqrt{\frac{7R}{5g}} \int_0^d \frac{dx}{\sqrt{(d^2 - x^2)}} = \frac{\pi}{2}\sqrt{\frac{7R}{5g}} = 24 \text{ minutes, } 59.627 \text{ seconds.}$$

Therefore the time is the same for any perfectly rough tunnel.

If the sphere starts with a velocity v_1^2, then for a perfectly smooth tunnel,

$$v^2 = \left(\frac{dx}{dt}\right)^2 = \frac{g}{R}(a^2 - x^2) + v_1^2,$$

supposing the sphere to start from A with the velocity v_1.

$$\therefore t = \sqrt{R}\int_0^a \frac{dx}{\sqrt{[Rv_1^2 + g(a^2 - x^2)]}} = \sqrt{\frac{R}{g}}\,\sin^{-1}\frac{a\sqrt{g}}{\sqrt{(Rv_1^2 + ga^2)}}.$$

This equals the time to go from A to C.

For a perfectly rough tunnel, $v^2=(dx/dt)^2=(5g/7R)(a^2-x^2)+v_1^2$.

$$\therefore t=\sqrt{7R}\int_0^a \frac{dx}{\sqrt{[7Rv_1^2+5g(a^2-x^2)]}}=\sqrt{\frac{7R}{5g}}\sin^{-1}\frac{a\sqrt{5g}}{\sqrt{(7Rv_1^2+5ga^2)}}.$$

III. If the tunnel extends beyond the surface of the earth: then let SB be the tunnel. Above the earth's surface, acceleration varies inversely as the square of the distance from the center.

Let $CO=c$, $OQ=y$, $CQ=x$, $SC=h$. Then $f:g=R^2:y^2$ or $f=gR^2/y^2=gR^2/(c^2+x^2)$. $f':f=\sin\theta:1$. $\therefore f'=f\sin\theta=fx/\sqrt{(c^2+x^2)}$.

$$\therefore f'=\frac{gR^2x}{\sqrt{[(c^2+x^2)^3]}}.\quad \therefore \frac{d^2x}{dt^2}+\frac{gR^2x}{\sqrt{[(c^2+x^2)^3]}}=0.$$

$$\therefore \left(\frac{dx}{dt}\right)^2=2gR^2\left[\frac{1}{\sqrt{(c^2+x^2)}}-\frac{1}{\sqrt{(c^2+h^2)}}\right]=v_1^2.$$

$$\therefore v_1=R\sqrt{2g\left[\frac{1}{\sqrt{(a^2+c^2)}}-\frac{1}{\sqrt{(h^2+c^2)}}\right]}=R\sqrt{\frac{2g}{R}-\frac{2g}{\sqrt{(c^2+h^2)}}}.$$

When the sphere reaches A,

$$t=\frac{\sqrt[4]{(c^2+h^2)}}{R\sqrt{(2g)}}\int_0^h\frac{\sqrt[4]{(c^2+x^2)}\,dx}{\sqrt{[\sqrt{(c^2+h^2)}-\sqrt{(c^2+x^2)}]}}.$$

$$\therefore t=\frac{2(c^2+h^2)^{\frac{5}{4}}}{R\sqrt{(2g)}}\int_{\phi_1}^{\frac{1}{2}\pi}\frac{\sin^4\psi\,d\psi}{\sqrt{[(c^2+h^2)\sin^4\psi-c^2]}},$$

where $c^2+x^2=(c^2+h^2)\sin^4\psi$ and $\phi_1=\sin^{-1}\sqrt[4]{\frac{R^2}{c^2+h^2}}$.

t=time to move from S to A. For a perfectly rough tunnel, $v_2=\sqrt{(\frac{5}{7})}v_1$; $t_1=\sqrt{(\frac{7}{5})}t$.

Also solved by S. Lefseletz.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

---

160. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that the integer next above $(1+\sqrt{3})^{2n}$ is divisible by 2^{n+1}.